Assignment 2

1. Comments on the rules

A. $\rightarrow$E (pg. 23) This rule is commonly known as Modus Ponens. This rule of inference says that from a conditional and the antecedent of that conditional, we can infer the consequent of that conditional. The $\rightarrow$E in SL corresponds to this rule of inference. It says that from sentences of the form $X \rightarrow Y$ and X, we can infer Y at a later line. Y will depend on all the assumptions on which X depends, plus all the assumptions on which $X \rightarrow Y$ depends.

In a correct use of $\rightarrow$E, the main connective must be a conditional, and the sentence X must be the exact antecedent of the conditional $X \rightarrow Y$.

EXAMPLE 1

1	(1)	$\sim\sim P$	A
2	(2)	$\sim\sim P \rightarrow (Q \rightarrow R)$	A
1,2	(3)	$Q \rightarrow R$	1,2 $\rightarrow$E

EXAMPLE 2

1	(1)	$((Q\rightarrow R)\rightarrow R)\rightarrow((R\rightarrow P)\rightarrow R)$	A
2	(2)	$(Q\rightarrow R)\rightarrow R$	A
1,2	(3)	$(R\rightarrow P)\rightarrow R$	1,2$\rightarrow$E

These are correct uses of the $\rightarrow$E rule. Notice that both the conditional and its antecedent may be as complicated as you please. Also, it doesn't matter in what order the conditional and its antecedent appear in the proof.

The following are incorrect uses of $\rightarrow$E:

EXAMPLE 3

1	(1)	P	A
2	(2)	$\sim(P\rightarrow Q)$	A
1,2	(3)	Q	1,2 $\rightarrow$E

EXAMPLE 4

1	(1)	$P\rightarrow(R\rightarrow Q)$	A
2	(2)	$R\rightarrow Q$	A
1,2	(3)	P	1,2 $\rightarrow$E

These are incorrect uses of $\rightarrow$E (and they are invalid arguments.) In example 3, the main connective of line 2 is the negation, not the arrow. In example 4, line 2 is the consequent of line 1, not the antecedent.

B. Modus Tollens (MT) (pg. 29) This rules says that from a conditional and the denial of its consequent, we can infer the denial of its antecedent.

EXAMPLE 5

1	(1)	$P\rightarrow(Q\rightarrow R)$	A
2	(2)	$\sim(Q\rightarrow R)$	A
1,2	(3)	$\sim P$	1,2 MT

EXAMPLE 6

1	(1)	$\sim(Q\rightarrow P)\rightarrow \sim(S\rightarrow R)$	A
2	(2)	$S\rightarrow R$	A
1,2	(3)	$Q\rightarrow P$	1,2 MT

Each of these is a correct use of MT. Again, notice that the conditional, its antecedent, and its consequent may be as complicated as you like. Also, note that the denial of a sentence can be its negation but it can also be the unnegation (the previous sentence was a negation and then we drop the '~').

The following are incorrect uses of MT:

EXAMPLE 7			EXAMPLE 8		
1	(1) P→(Q→R)	A	1	(1) ~P→(Q→R)	A
2	(2) ~R	A	2	(2) P	A
1,2	(3) ~P	1,2 MT	1,2	(3) ~(Q→R)	1,2 MT

Both of these are incorrect uses of MT and are in fact invalid arguments. In example 7, ~R is not the negation of the entire consequent, but only part of it. In example 8, P is the denial of the antecedent of the conditional, not its consequent.

C. <u>→I</u> (pg. 22) This rule is commonly known as conditional proof. According to this rule, if X is introduced as an assumption and Y is derived at a later line, then we may infer X→Y at a later line. X→Y will depend on the same assumptions that Y depends on **<u>minus</u>** the assumption X.

EXAMPLE 9			EXAMPLE 10		
1	(1) P→ ~Q	A	1	(1) P→(Q→R)	A
2	(2) R → Q	A	2	(2) P	A
3	(3) P	A	1,2	(3) Q→R	1,2 →E
1,3	(4) ~Q	1,3 →E	4	(4) ~R	A
1,2,3	(5) ~R	2,4 MT	1,2,4	(5) ~Q	3,4 MT
1,2	(6) P→ ~R	5 →I (3)	4	(6) (Q→R)→ ~Q	5 →I (3)

Example 9 is a correct use of →I. Example 10 is <u>not</u> a correct use of →I. In this example, line 3 is not an assumption and so cannot be discharged by use of the →I rule.

2. Strategy for proving sequents

A good general problem-solving strategy is to work from the ends of a problem, from both what you know and from your goal, and end up in the middle. Therefore, you should always look at what you want to prove to help you determine how you should go about proving it. If your goal is a conditional of the form X → Y, you should introduce X as an assumption and try to prove Y. If you succeed in proving Y, you can then derive X → Y by the →I rule. If Y itself is a conditional, say Z → W, you should repeat the strategy by assuming Z and trying to prove W.

If what you are trying to prove is not a conditional, you will have to get it by →E or by MT (for this assignment). For instance, if you want to prove P and you notice that you have (Q→R)→P earlier in your proof, a good strategy is to try to derive Q→R and then use →E to get P. Since Q→R is a conditional, you should assume Q and try to derive R.

EXAMPLE 11 $P \to Q \vdash (R \to P) \to (R \to Q)$

Step 1. Since our goal, $(R\to P)\to(R\to Q)$ is a conditional, I will assume $R\to P$ and try to prove $R\to Q$.	1	(1)	$P\to Q$	A
	2	(2)	$R\to P$	A
	1,2	(n-1)	$R\to Q$	new goal
	1	(n)	$(R\to P)\to(R\to Q)$	$\to$I

	1	(1)	$P\to Q$	A
Step 2. My new goal is to prove $R\to Q$. Since this is a conditional, I will assume R and try to prove Q.	2	(2)	$R\to P$	A
	3	(3)	R	A
		(n-2)	Q	new goal
	1,2	(n-1)	$R\to Q$	$\to$I
	1	(n)	$(R\to P)\to(R\to Q)$	$\to$I

Step 3. Now my goal is to prove Q. So I see what I already have in the proof and look for uses of $\to$E and MT.	1	(1)	$P\to Q$	A
	2	(2)	$R\to P$	A
	3	(3)	R	A
	2,3	(4)	P	2,3 $\to$E
	1,2,3	(5)	Q	1,4 $\to$E
	1,2	(6)	$R\to Q$	5 $\to$I (3)
	1	(7)	$(R\to P)\to(R\to Q)$	6$\to$I (2)

EXAMPLE 12 $P\to(Q\to R) \vdash \sim R\to(P\to\sim Q)$

Step 1. Since what I wish to prove is a conditional, I assume its antecedent and try to prove its consequent.	1	(1)	$P\to(Q\to R)$	A
	2	(2)	$\sim R$	A
	1,2	(n-1)	$P\to\sim Q$	new goal
	1	(n)	$\sim R\to(P\to\sim Q)$	$\to$I

	1	(1)	$P\to(Q\to R)$	A
Step 2. Since my new goal is a conditional, I will assume its antecedent and try to prove its consequent.	2	(2)	$\sim R$	A
	3	(3)	P	A
	1,2,3	(n-2)	$\sim Q$	new goal
	1,2	(n-1)	$P\to\sim Q$	$\to$I (3)
	1	(n)	$\sim R\to(P\to\sim Q)$	$\to$I (2)

Step 3. I will now attempt to prove $\sim Q$ from what I have.	1	(1)	$P\to(Q\to R)$	A
	2	(2)	$\sim R$	A
	3	(3)	P	A
	1,3	(4)	$Q\to R$	1,3 $\to$E
	1,2,3	(5)	$\sim Q$	2,4 MT
	1,2	(6)	$P\to\sim Q$	5 $\to$I (3)
	1	(7)	$\sim R\to(P\to\sim Q)$	6 $\to$I (2)

EXAMPLE 13 $(Q\to R)\to P,\ S\to R \vdash (Q\to S)\to P$

Step 1. Since our goal is a conditional, I will assume its antecedent and try to prove its consequent.

1	(1)	$(Q\to R)\to P$	A
2	(2)	$S\to R$	A
3	(3)	$Q\to S$	A
1,2,3	(n-1)	P	new goal
1,2	(n)	$(Q\to S)\to P$	$\to$I (3)

Step 2. P, the new goal, is not a conditional. I should use $\to$E and MT on what I have to see if I can make some progress. But right now I can't use either of those rules. But by examining what I do have (lines 1-3) I see that P could be obtained from line 1 if I could prove $Q\to R$, So my new goal is now $Q\to R$.

1	(1)	$(Q\to R)\to P$	A
2	(2)	$S\to R$	A
3	(3)	$Q\to S$	A
	(n-2)	$Q\to R$	new goal
1,2,3	(n-1)	P	$\to$E
1,2	(n)	$(Q\to S)\to P$	$\to$I (3)

Step 3. Since $Q\to R$, my new goal, is a conditional, I will assume its antecedent and try to prove its consequent.

1	(1)	$(Q\to R)\to P$	A
2	(2)	$S\to R$	A
3	(3)	$Q\to S$	A
4	(4)	Q	A
	(n-3)	R	new goal
	(n-2)	$Q\to R$	$\to$I (4)
1,2,3	(n-1)	P	$\to$E
1,2	(n)	$(Q\to S)\to P$	$\to$I (3)

Step 4. The new goal, R, is not a conditional. But since I now have an additional line to work with, I can make progress using $\to$E and MT. We can get R this way without too much trouble.

1	(1)	$(Q\to R)\to P$	A
2	(2)	$S\to R$	A
3	(3)	$Q\to S$	A
4	(4)	Q	A
3,4	(5)	S	3,4 $\to$E
2,3,4	(6)	R	2,5 $\to$E
2,3	(7)	$Q\to R$	6 $\to$I (4)
1,2,3	(8)	P	1,7 $\to$E
1,2	(9)	$(Q\to S)\to P$	8 $\to$I (3)